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A COMPUTERIZED SYMBOLIC INTEGRATION TECHNIQUE FOR DEVELOPMENT OF TRIANGULAR AND QUADRILATERAL COMPOSITE SHALLOW-SHELL FINITE ELEMENTS

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The integrals are performed symbolically by using the symbolic-and-algebraic-manipulation language MACSYMA. The efficiency of using symbolic integration in the element development is demonstrated by comparing the number of floating-point arithmetic operations required in this approach with those required by a commonly used numerical quadrature technique.

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SUMMARY

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INTRODUCTION

Although considerable literature has been devoted to the analysis of isotropic shells using doubly curved finite elements, investigations of the finite-element analysis of laminated composite shells are rather limited in extent. The reliable prediction of the response of composite shells often requires the use of high-order shear-flexible elements with the consequent increase in the developmental effort of the elemental matrices (the stiffness, geometric stiffness, and mass matrices).

In most of the published work, numerical integration was used for the evaluation of the element stiffness matrices. Although numerical integration is both simple and adaptable to computer programing, it can become computationally very expensive, particularly for higher order elements. This drawback has been recognized and improvements have been suggested (refs. 1 and 2); however, the difficulty has not been overcome. This raises the question as to whether an alternative approach such as using analytic (closed form or symbolic) integration in the element development is computationally more efficient. The present study addresses this question. More specifically, the objective of this paper is to demonstrate the computational advantages resulting from the use of computerized symbolic integration in conjunction with group-theoretic techniques (or symmetry transformations) in the development of triangular and quadrilateral shallow shell elements.

The analytical formulation is based on a form of the shallow-shell theory modified such that the effects of shear deformation and rotary inertia are included. Indicial notation is used throughout the development, since it is particularly useful in identifying the symmetries. The integrals are performed symbolically by using the symbolic-and-algebraic-manipulation language MACSYMA[1] (refs. 3 and 4). This is done in order to reduce the tedium of analysis and to decrease the likelihood of errors. The use of group-theoretic techniques (or symmetry transformations) results in considerable reduction in both the amount of analytic computation needed and the size of the programs for numerical computation.

SYMBOLS

$A^{ijk\ell}, B_{\alpha}^{ijk}, C_{\alpha\beta}^{ij}$ basic integrals defined in equations (14) to (16)

a^n, d^n, e^n representations of dihedral groups defined in equations (56), (60), (61), and (69)

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[1]The MACSYMA system is being developed by the Mathlab group at Massachusetts Institute of Technology under the support of ARPA (Advanced Research Projects Agency of the U.S. Department of Defense) through Office of Naval Research Contract No. N00014-70-A-0362-001.

$[C]$ 8 by 8 matrix of shell stiffnesses

$C_{\alpha\beta\gamma\rho}, C_{\alpha 3\beta 3}$ extensional and transverse shear stiffnesses of shell, respectively

$c^{(k)}_{\alpha\beta\gamma\rho}, c^{(k)}_{\alpha 3\beta 3}$ extensional and transverse shear stiffnesses of kth layer of shell

c_σ portion of shell boundary over which tractions are prescribed

$D_{\alpha\beta\gamma\rho}$ bending stiffnesses of shell

E_L, E_T elastic moduli in direction of fibers and normal to fibers, respectively

$e_{\alpha\beta}$ permutation symbol

$F_{\alpha\beta\gamma\rho}$ stiffness interaction coefficients of shell

$\left.\begin{matrix} \mathcal{F}^{ij}_{\alpha\beta 0}, \mathcal{F}^{ij}_{\alpha\beta 1}, \\ \mathcal{F}^{ij}_{\alpha\beta 2}, \mathcal{F}^{ij}_{\alpha\beta 3} \end{matrix}\right\}$ functions of coordinates of corner nodes

G_{LT}, G_{TT} shear moduli in plane of fibers and normal to plane of fibers, respectively

$\bar{\mathcal{G}}^{ij}_{\alpha\beta 0}, \bar{\mathcal{G}}^{ij}_{\alpha\beta 1}$ functions of coordinates of corner nodes of elements

H_ℓ weighting coefficients for numerical quadrature

h_k, h_{k-1} distances from reference (middle) surface to top and bottom surfaces of kth layer, respectively

$[K]$ element stiffness matrix

K^{ij}_{IJ} stiffness coefficients of shell element

$[\overline{K}]$ geometric stiffness matrix

$\overline{K}_{IJ}^{ij}$ geometric stiffness coefficients of shell element

$k_{\alpha\beta}$ curvatures and twist of shell reference surface

$k_{\alpha\beta}^{i}$ nodal values of $k_{\alpha\beta}$

$L_1, L_2, \overline{L}_1, \overline{L}_2, \overline{L}$ logarithmic functions defined in equations (33), (38), and (42)

$[M]$ consistent mass matrix

M_{IJ}^{ij} consistent mass coefficients of shell element

$\tilde{\mathcal{M}}_{\alpha\beta}$ prescribed bending stress resultants on shell boundary

m_0, m_1, m_2 density parameters of shell

N^i shape or interpolation function

$\tilde{\mathcal{N}}_{\alpha\beta}$ prescribed extensional (in-plane) stress resultants on shell boundary

$\mathcal{N}_{\alpha\beta}^{o}$ relative magnitudes of initial stress resultants (prestress components)

n_α unit outward normal to shell boundary

$\{P\}$ consistent load vector

P_I^i consistent load coefficients

p_α, p external load intensities in coordinate directions

p_α^i, p^i nodal values of p_α and p

$\tilde{\mathcal{Q}}_\beta$ prescribed transverse shear stress resultants on shell boundary

$R_1^{ijk\ell}, R_2^{ijk\ell}, R_3^{ijk\ell}$ coefficients associated with A-integrals

$\mathcal{R}_1^m, \mathcal{R}_2^m, \mathcal{R}_3^m, \mathcal{R}_4^m$ integers associated with representative A-integrals

r number of shape functions associated with an element

$S_1^{ijk}, S_2^{ijk}, S_3^{ijk}$ coefficients associated with B-integrals

$\mathcal{S}_1^{\overline{m}}, \mathcal{S}_2^{\overline{m}}, \mathcal{S}_3^{\overline{m}}, \mathcal{S}_4^{\overline{m}}$ integers associated with representative B-integrals

s ratio, U_2/U_1

$\tilde{s}$ a quantity defined in equation (39)

T kinetic energy of shell

T transpose symbol

$T_1^{ij}, T_2^{ij}, T_3^{ij}, T_4^{ij}$ coefficients associated with C-integrals

$\mathcal{T}_1^m, \mathcal{T}_2^m, \mathcal{T}_3^m, \mathcal{T}_4^m, \mathcal{T}_5^m$ integers associated with representative C-integrals

t ratio, U_3/U_1

$\tilde{t}$ a quantity defined in equation (39)

U strain energy of shell

U^o strain energy due to prestress

U_1, U_2, U_3 functions of nodal coordinates

u_α, w displacement components in coordinate directions (see fig. 1)

$V_{\alpha 1}, V_{\alpha 2}, V_{\alpha 3}$
$V^{\bar{\bar{n}}}_{\alpha 1}, V^{\bar{\bar{n}}}_{\alpha 2}, V^{\bar{\bar{n}}}_{\alpha 3}$ linear functions of coordinates of corner nodes

W work done by external forces

x_α, x_3 orthogonal coordinate system (see fig. 1)

x^i_α values of x_α for corner nodes

λ in-plane load parameter

μ number of numerical quadrature points

ν_{LT} Poisson's ratio measuring strain in T direction due to uniaxial normal stress in the L direction

ξ_α local dimensionless coordinates in the shell domain

$\Pi(u_\alpha, w, \phi_\alpha)$ functional defined in equation (1)

$\rho_s^{(k)}$ density of kth layer of shell

σ, τ permutations

ϕ_α rotation components

$\{\psi\}$ nodal displacement vector

ψ^j_J nodal displacement parameters

Ω shell domain

$\Omega^{(e)}$ shell element domain

ω circular frequency of vibration of shell

∂_α $= \partial/\partial x_\alpha$

The range of the different indices is as follows:

Lower case Latin indices: $1 \rightarrow r$

Upper case Latin indices: $1 \rightarrow 5$

Greek indices: 1,2

The finite-element models are designated:

SQr stiffness formulation, quadrilateral element, r shape functions per fundamental unknown

STr stiffness formulation, triangular element, r shape functions per fundamental unknown

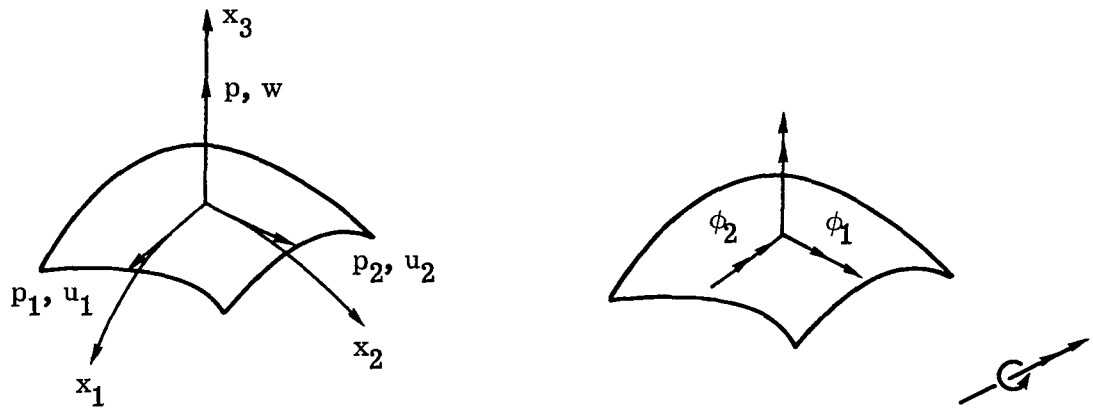
The groups are designated:

$\mathcal{D}_3$ six-element dihedral group

$\mathcal{D}_4$ eight-element dihedral group

FINITE-ELEMENT FORMULATION

The analytical formulation is based on a form of the shallow-shell theory with the effects of shear deformation, anisotropic material behavior, rotary inertia, and bending-extensional coupling included. (See ref. 5.) The prebuckling state of the shell is assumed to be a momentless state. A displacement (stiffness) formulation is used in which the fundamental unknowns (dependent variables) consist of five generalized displacements – the two tangential displacement components u_α, the normal displacement w, and the two rotation components ϕ_α. (See fig. 1.) Throughout this paper the range of the Greek indices is 1,2 and it is implied that any index Greek or Latin, which appears twice in the same term is to be summed over.



(a) Displacements and loads. (b) Rotations.

Figure 1.- Shell element and sign convention.

Governing Functional of Shell

The fundamental unknowns are assumed to vary sinusoidally with angular velocity ω (the circular frequency of vibration of the shell). The functional used in the development of the elemental matrices is given (see ref. 6) by

$$\Pi(u_\alpha, w, \phi_\alpha) = U + U^O - W - T \tag{1}$$

where U and U^O are the strain energies due to deformation and prestress, W is the potential energy of external forces, and T is the kinetic energy of the shell.

The expressions of U, U^O, W, and T in terms of the generalized displacements are

$$U = \frac{1}{2}\int_\Omega \Big\{ C_{\alpha\beta\gamma\rho}\Big[\partial_\alpha u_\beta\, \partial_\gamma u_\rho + 2k_{\alpha\beta}\, \partial_\gamma u_\rho\, w + k_{\alpha\beta}\, k_{\gamma\rho}\, (w)^2\Big] + 2F_{\alpha\beta\gamma\rho}\Big[\partial_\alpha u_\beta\, \partial_\gamma \phi_\rho + k_{\alpha\beta}\, \partial_\gamma \phi_\rho\, w\Big] + D_{\alpha\beta\gamma\rho}\, \partial_\alpha \phi_\beta\, \partial_\gamma \phi_\rho + C_{\alpha 3 \beta 3}\Big[\partial_\alpha w\, \partial_\beta w + 2\phi_\alpha\, \partial_\beta w + \phi_\alpha \phi_\beta\Big]\Big\}\, d\Omega \tag{2}$$

$$U^O = \frac{\lambda}{2}\int_\Omega \mathcal{N}^O_{\alpha\beta}\, \partial_\alpha w\, \partial_\beta w\, d\Omega \tag{3}$$

$$W = \int_\Omega (p_\alpha u_\alpha + pw)\, d\Omega + \int_{c_\sigma} \left(\tilde{\mathcal{N}}_{\alpha\beta} u_\alpha + \tilde{\mathcal{M}}_{\alpha\beta}\phi_\alpha + \tilde{\mathcal{Q}}_\beta w\right) n_\beta\, dc \tag{4}$$

and

$$T = \frac{\omega^2}{2}\int_{\Omega}\left[m_0\left(u_\alpha u_\alpha + (w)^2\right) + 2m_1 u_\alpha \phi_\alpha + m_2 \phi_\alpha \phi_\alpha\right] d\Omega \tag{5}$$

where $C_{\alpha\beta\gamma\rho}$, $D_{\alpha\beta\gamma\rho}$, and $F_{\alpha\beta\gamma\rho}$ are the extensional, bending, and stiffness interaction coefficients of the shell; $C_{\alpha 3\beta 3}$ are the transverse shear stiffnesses of the shell (see appendix A); $k_{\alpha\beta}$ are the curvature components and twist of the shell surface; $\lambda\mathcal{N}^0_{\alpha\beta}$ are the initial stress resultants (prestress field), λ is the in-plane load parameter; p_α and p are the external load components in the coordinate directions x_α and x_3, respectively; m_0, m_1, and m_2 are density parameters of the shell defined in appendix A; $\tilde{\mathcal{N}}_{\alpha\beta}$, $\tilde{\mathcal{M}}_{\alpha\beta}$, and $\tilde{\mathcal{Q}}_\beta$ are prescribed extensional, bending, and transverse shear stress resultants, respectively; Ω is the shell domain; c_σ is the portion of the boundary over which tractions are prescribed; and n_α is the unit outward normal to the boundary.

Finite-Element Discretization

The shell domain Ω is partitioned into triangular or quadrilateral subdomains $\Omega^{(e)}$. With each $\Omega^{(e)}$ is associated a finite element for which the fundamental unknowns are approximated by expressions of the form

$$\left.\begin{aligned} u_\alpha &= N^i \psi^i_\alpha \\ w &= N^i \psi^i_3 \\ \phi_\alpha &= N^i \psi^i_{3+\alpha} \end{aligned}\right\} \tag{6}$$

where N^i are the shape (or interpolation) functions; ψ^i_J $(J = 1 \rightarrow 5)$ are displacement parameters; r equals the number of shape functions in the approximation; and a repeated lower case Latin index denotes summation over the range $1 \rightarrow r$. Each set of displacement parameters $\psi^i_1, \psi^i_2, \ldots, \psi^i_5$ is either associated with a node of the element or with a bubble function.

The curvatures $k_{\alpha\beta}$ and the external load components p_α and p are approximated by the same shape functions used in approximating the generalized displacements, that is,

$$\left.\begin{aligned} k_{\alpha\beta} &= N^i k^i_{\alpha\beta} \\ p_\alpha &= N^i p^i_\alpha \\ p &= N^i p^i \end{aligned}\right\} \tag{7}$$

where $k^i_{\alpha\beta}$, p^i_α, and p^i are the nodal values of $k_{\alpha\beta}$, p_α, and p, respectively. To simplify the development of the elemental matrices, it is assumed in subsequent sections that the shell stiffnesses C's, F's, and D's as well as the density parameters m_0, m_1, m_2, and the initial stress resultants are constants within each element. Equations (2) to (5) can be expressed in terms of the shape functions N^i and the nodal parameters ψ^i_J as follows (for simplicity, the line integrals in eq. (r) are assumed to make no contribution):

$$\begin{aligned} U = \frac{1}{2} \sum_{\text{Elements}} \int_{\Omega(e)} \Big[& C_{\alpha\beta\gamma\rho} \Big(\partial_\alpha N^i \, \partial_\gamma N^j \, \psi^i_\beta \psi^j_\rho + 2k^n_{\alpha\beta} \, \partial_\gamma N^i \, N^j N^n \psi^i_\rho \psi^j_3 \\ & + k^n_{\alpha\beta} \, k^\ell_{\gamma\rho} \, N^i N^j N^n N^\ell \psi^i_3 \psi^j_3 \Big) + F_{\alpha\beta\gamma\rho} \Big(\partial_\alpha N^i \, \partial_\gamma N^j \, \psi^i_\beta \psi^j_{\rho+3} + k^n_{\alpha\beta} \, \partial_\gamma N^i \, N^j N^n \psi^i_{\rho+3} \, \psi^j_3 \Big) \\ & + D_{\alpha\beta\gamma\rho} \, \partial_\alpha N^i \, \partial_\gamma N^j \, \psi^i_{\beta+3} \, \psi^j_{\rho+3} + C_{\alpha 3 \beta 3} \Big(\partial_\alpha N^i \, \partial_\beta N^j \, \psi^i_3 \psi^j_3 + 2N^i \, \partial_\beta N^j \, \psi^i_{\alpha+3} \, \psi^j_3 \\ & + N^i N^j \psi^i_{\alpha+3} \, \psi^j_{\beta+3} \Big) \Big] \, d\Omega \end{aligned} \tag{8}$$

$$U^o = \frac{\lambda}{2} \sum_{\text{Elements}} \int_{\Omega(e)} \mathcal{N}^o_{\alpha\beta} \, \partial_\alpha N^i \, \partial_\beta N^j \, \psi^i_3 \psi^j_3 \, d\Omega \tag{9}$$

$$W = \sum_{\text{Elements}} \int_{\Omega(e)} N^i N^j \left(p^i_\alpha \psi^j_\alpha + p^i \psi^j_3 \right) d\Omega \tag{10}$$

$$T = \frac{\omega^2}{2} \sum_{\text{Elements}} \int_{\Omega(e)} N^i N^j \Big[m_0 \left(\psi^i_\alpha \psi^j_\alpha + \psi^i_3 \psi^j_3 \right) + 2m_1 \psi^i_\alpha \psi^j_{3+\alpha} + m_2 \psi^i_{3+\alpha} \, \psi^j_{3+\alpha} \Big] \, d\Omega \tag{11}$$

The governing equations for each element are obtained by applying the stationary conditions for the functional in equation (1). This leads to the following set of equations:

$$\left(K_{IJ}^{ij} + \lambda \bar{K}_{IJ}^{ij}\right)\psi_J^j = P_I^i + \omega^2 M_{IJ}^{ij}\,\psi_J^j \tag{12}$$

where K_{IJ}^{ij} are the stiffness coefficients; $\bar{K}_{IJ}^{ij}$ are the geometric stiffness (prestress) coefficients; M_{IJ}^{ij} and P_I^i are consistent mass and load coefficients; and λ is an in-plane loading parameter used to determine the buckling load or the magnitude of the prestress. The formulas for the aforementioned stiffness, mass and load coefficients are given in appendix B. For static stress analysis problems, $\lambda = \omega = 0$; and for free vibration problems, $\lambda = 0$ and $P_I^i = 0$. The K's, $\bar{K}$'s, and M's are completely symmetric under the interchange of one pair of indices for another, each pair of indices consisting of a superscript and the subscript just beneath it.

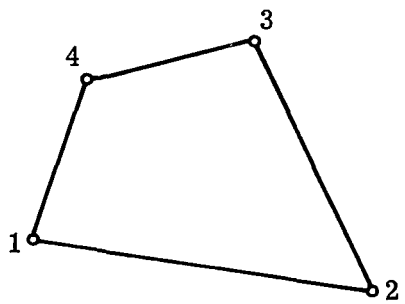
To write equation (12) in matrix form, the first superscript-subscript pair of each of the K's, $\bar{K}$'s, and M's defines the row number and the second pair defines the column number. For example, the term K_{IJ}^{ij} will be located in the $[5(i - 1) + I]$th row and in the $[5(j - 1) + J]$th column of the element stiffness matrix. Similarly, the P's and ψ's can be written in column vector form. Thus, equation (12) becomes

$$\left([K] + \lambda[\bar{K}]\right)\{\psi\} = \{P\} + \omega^2[M]\{\psi\} \tag{13}$$

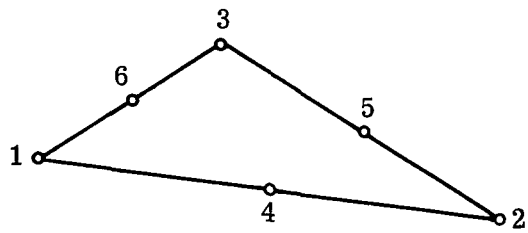
Representation of Fundamental Unknowns

Two triangular and four quadrilateral conforming elements were developed for the present study. The nodes of these elements are shown in figure 2, and their characteristics are summarized in table I. The triangular elements developed have six or ten nodes and the quadrilateral elements have four or eight nodes. For the two triangular elements and for two of the quadrilateral elements the range r of the superscripts in equations (6) and (7) is the number of nodes (that is, 6, 10, 4, or 8). For the two remaining quadrilateral elements, r is one greater than the number of nodes (that is, r = 5 or 9). The additional shape function, which vanishes along the entire element boundary and is not associated with a node, is usually referred to as a "bubble function."

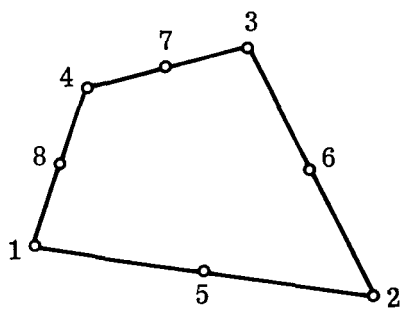
Since for each element all five of the fundamental unknowns are approximated by the same set of shape functions (see eq. (6)), the number of degrees of freedom per element is 5r and thus for the elements developed the element stiffness matrices [K] of equation (13) range in size from 20 by 20 (for SQ4) to 50 by 50 (for ST10).



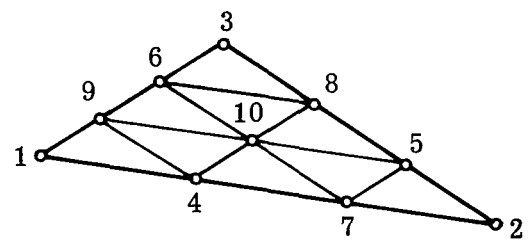
(a) SQ4 and SQ5.



(b) ST6.



(c) SQ8 and SQ9.



(d) ST10.

Figure 2.- The finite elements used in the present study.

The nodal parameters ψ_I^i introduced in equation (6) were selected to be values of the fundamental unknowns at the different nodes, and the approximations to the fundamental unknowns are continuous across the element boundaries.

COMPUTATION OF ELEMENT CHARACTERISTICS

Each component of the arrays K_{IJ}^{ij}, $\bar{K}_{IJ}^{ij}$, M_{IJ}^{ij}, and P_I^i is a linear combination of integrals whose integrands are products of shape functions and their derivatives. (See appendix B.) If a constant shape function $N^0 = 1$ is introduced, then the integrals in appendix B can be reduced to linear combinations of integrals of the following three types:

$$A^{ijk\ell} = \int_{\Omega(e)} N^i N^j N^k N^\ell \, d\Omega \qquad (i, j, k, \ell = 0 \rightarrow r) \tag{14}$$

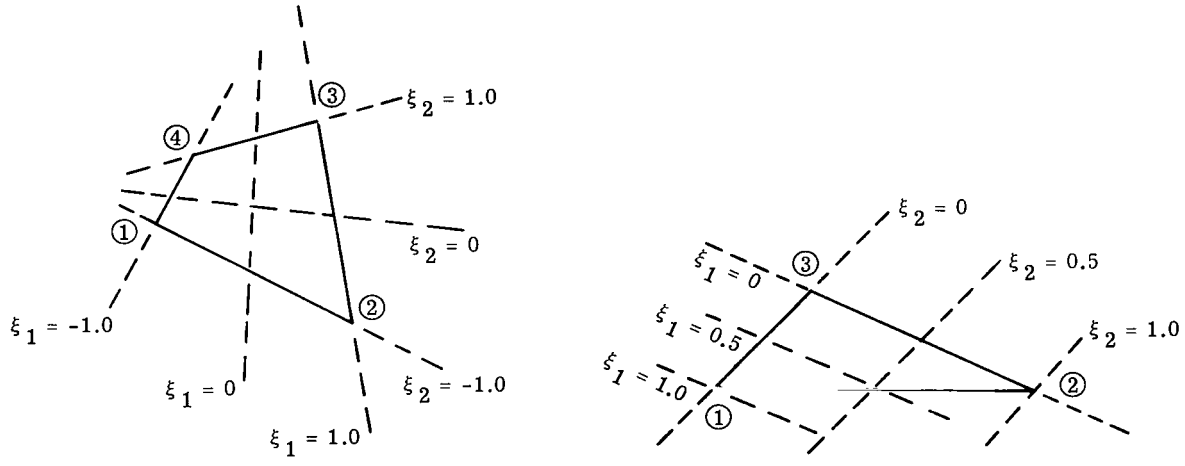
$$B_\alpha^{ijk} = \int_{\Omega(e)} N^i N^j \, \partial_\alpha N^k \, d\Omega \qquad (i, j = 0 \rightarrow r; \quad k = 1 \rightarrow r) \tag{15}$$

$$C^{ij}_{\alpha\beta} = \int_{\Omega(e)} \partial_\alpha N^i \, \partial_\beta N^j \, d\Omega \qquad (i, j = 1 \rightarrow r) \qquad (16)$$

The evaluation of the stiffness, mass, and load coefficients can be divided into two separate problems: (a) the determination of the coefficients in the linear combinations and (b) the evaluation of the A-, B-, and C-integrals. Since problem (a) is relatively simple, the present study focuses on simplifications and means of reducing the effort involved in problem (b).

EVALUATION OF INTEGRALS

For the evaluation of the A-, B-, and C-integrals (eqs. (14) to (16)), it is convenient to introduce a dimensionless local coordinate system ξ_α ($\alpha = 1,2$) in terms of which the limits of integration are simply expressed. For triangular and quadrilateral elements, the ξ_α are chosen to be the area and quadrilateral coordinates, respectively. (See ref. 7 and fig. 3.) In the present study, the shape functions are simply polynomials in ξ_α (see appendix C) and the Cartesian coordinates x_α are linear or bilinear functions of ξ_α.



(a) Quadrilateral. (b) Triangular.

Figure 3.- Local coordinate systems for quadrilateral and triangular finite elements.

The relations between the Cartesian and local dimensionless coordinates are

$$d\Omega = \frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)} \, d\xi_1 \, d\xi_2 \qquad (17)$$

and

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial \xi_\gamma}{\partial x_\alpha} \frac{\partial}{\partial \xi_\gamma} = \left[\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}\right]^{-1} e_{\alpha\beta}\, e_{\gamma\rho} \frac{\partial x_\beta}{\partial \xi_\rho} \frac{\partial}{\partial \xi_\gamma} \tag{18}$$

where $\partial(x_1,x_2)/\partial(\xi_1,\xi_2)$ is the Jacobian determinant and $\partial\xi_\gamma/\partial x_\alpha$ is the inverse of the Jacobian matrix. See ref. 8 for more explicit expressions for $\partial(x_1,x_2)/\partial(\xi_1,\xi_2)$ and $\partial/\partial x_\alpha$ for selected cases.

Through the use of equations (17) and (18), the A-, B-, and C-integrals can be expressed in terms of the local coordinates ξ_α as

$$A^{ijk\ell} = \int_{\Omega(e)} N^i N^j N^k N^\ell \frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}\, d\xi_1\, d\xi_2 \tag{19}$$

$$B_\alpha^{i\,jk} = e_{\alpha\beta}\, e_{\gamma\rho} \int_{\Omega(e)} N^i N^j \frac{\partial x_\beta}{\partial \xi_\rho} \frac{\partial N^k}{\partial \xi_\gamma}\, d\xi_1\, d\xi_2 \tag{20}$$

$$C_{\alpha\beta}^{i\,j} = e_{\alpha\gamma}\, e_{\beta\rho}\, e_{\kappa\lambda}\, e_{\mu\nu} \int_{\Omega(e)} \frac{\partial N^i}{\partial \xi_\kappa} \frac{\partial N^j}{\partial \xi_\mu} \frac{\partial x_\gamma}{\partial \xi_\lambda} \frac{\partial x_\rho}{\partial \xi_\nu} \left[\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}\right]^{-1} d\xi_1\, d\xi_2 \tag{21}$$

The A-integrals are completely symmetric under interchange of indices, and the B- and C-integrals satisfy the following symmetry relations:

$$\left.\begin{aligned} B_\alpha^{i\,jk} &= B_\alpha^{j\,ik} \\ C_{\alpha\beta}^{i\,j} &= C_{\beta\alpha}^{j\,i} \end{aligned}\right\} \tag{22}$$

Thus, the number of numerically distinct A-, B-, and C-integrals in equations (19) to (21) is $(r+4)!/(4!r!)$, $\frac{1}{2}(r+1)(r+2)(2r)$, and $\frac{1}{2}(2r)(2r+1)$, respectively. (See ref. 9 and table II.)

When the integrands of equations (19) to (21) are polynomials in ξ_α, the analytic (symbolic) evaluation of the corresponding integrals is a straightforward task. Examination of equations (19) and (20) reveals that this is the case for all the A- and B-integrals. The integrands of the C-integrals (eq. (21)) are polynomials in ξ_α only when the Jaco-

bian determinant $\partial(x_1,x_2)/\partial(\xi_1,\xi_2)$ is independent of ξ_α, and this is the case for triangular and parallelogram elements. For a general quadrilateral element, the Jacobian determinant is linear in ξ_α, and consequently, the C-integrals are complicated functions of the nodal coordinates and involve logarithms.

The forms of the analytic expressions for the A-, B-, and C-integrals for both the triangular and quadrilateral elements are given subsequently.

Triangular Elements

For triangular finite elements, the A-, B-, and C-integrals may be expressed as

$$A^{ijk\ell} = R_1^{ijk\ell}\, U_1 \tag{23}$$

$$B_\alpha^{ijk} = \begin{bmatrix} S_1^{ijk} & S_2^{ijk} \end{bmatrix} \begin{bmatrix} V_{\alpha 1} \\ V_{\alpha 2} \end{bmatrix} \tag{24}$$

$$C_{\alpha\beta}^{ij} = \frac{1}{U_1} \begin{bmatrix} V_{\alpha 1} & V_{\alpha 2} \end{bmatrix} \begin{bmatrix} T_1^{ij} & T_2^{ij} \\ T_3^{ij} & T_4^{ij} \end{bmatrix} \begin{bmatrix} V_{\beta 1} \\ V_{\beta 2} \end{bmatrix} \tag{25}$$

where the R's, S's, and T's are rational numbers (independent of the nodal coordinates) and are discussed later; the V's are functions of the nodal coordinates given by

$$\left.\begin{aligned} V_{\alpha 1} &= e_{\alpha\beta}\left(x_\beta^1 - x_\beta^2\right) \\ V_{\alpha 2} &= e_{\alpha\beta}\left(x_\beta^2 - x_\beta^3\right) \end{aligned}\right\} \tag{26}$$

The quantity U_1 is the area of the triangular element and is given by

$$U_1 = \frac{1}{2}\, e_{\alpha\beta}\, V_{\alpha 1} V_{\beta 2} \tag{27}$$

and x_β^1, x_β^2, and x_β^3 are the coordinates of the corner nodes numbered as in figure 2(b).

Since the five quantities U_1, $V_{\alpha 1}$, and $V_{\alpha 2}$ are functions of the nodal coordinates, they need to be evaluated for each element. On the other hand, the R's, S's, and

T's are the same for all triangular elements having the same number of nodes. As will be demonstrated in subsequent sections, the evaluation of the A-, B-, and C-integrals using equations (23) to (25) is computationally more efficient than using a numerical quadrature technique wherein the integrands have to be evaluated at each of the quadrature points, multiplied by the weighting coefficients, and then summed.

Quadrilateral Elements

For quadrilateral elements, the A-, B-, and C-integrals may be expressed as

$$A^{ijk\ell} = \begin{bmatrix} R_1^{ijk\ell} & R_2^{ijk\ell} & R_3^{ijk\ell} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{28}$$

$$B_\alpha^{ijk} = \begin{bmatrix} S_1^{ijk} & S_2^{ijk} & S_3^{ijk} \end{bmatrix} \begin{bmatrix} V_{\alpha 1} \\ V_{\alpha 2} \\ V_{\alpha 3} \end{bmatrix} \tag{29}$$

$$C_{\alpha\beta}^{ij} = \frac{1}{U_1} \left\{ \mathcal{T}_{\alpha\beta 0}^{ij} + \begin{bmatrix} \mathcal{T}_{\alpha\beta 1}^{ij} & \mathcal{T}_{\alpha\beta 2}^{ij} & \mathcal{T}_{\alpha\beta 3}^{ij} \end{bmatrix} \begin{bmatrix} L_1(s,t) \\ L_2(s,t) \\ L_2(t,s) \end{bmatrix} \right\} \tag{30}$$

where the R's and S's are rational numbers (independent of the nodal coordinates) and are discussed later; the U's and V's are functions of the nodal coordinates given by

$$\left.\begin{aligned} V_{\alpha 1} &= \frac{e_{\alpha\beta}\left(x_\beta^1 - x_\beta^2 + x_\beta^3 - x_\beta^4 \right)}{4} \\ V_{\alpha 2} &= \frac{e_{\alpha\beta}\left(-x_\beta^1 + x_\beta^2 + x_\beta^3 - x_\beta^4 \right)}{4} \\ V_{\alpha 3} &= \frac{e_{\alpha\beta}\left(-x_\beta^1 - x_\beta^2 + x_\beta^3 + x_\beta^4 \right)}{4} \end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned} U_1 &= e_{\alpha\beta}\, V_{\alpha 2} V_{\beta 3} \\ U_2 &= e_{\alpha\beta}\, V_{\alpha 3} V_{\beta 1} \\ U_3 &= e_{\alpha\beta}\, V_{\alpha 1} V_{\beta 2} \end{aligned}\right\} \tag{32}$$

and x_α^1, x_α^2, x_α^3, and x_α^4 are the coordinates of the corner nodes numbered as in figure 2(a). The quantity U_1 is equal to one-fourth the area of the element.

The $\mathcal{F}$'s in equation (30) are complicated rational functions of s, t, and the V's, and the L's are logarithmic functions given by

$$\left.\begin{aligned} L_1(s,t) = L\ (t,s) &= \frac{1}{2} \log \left[\frac{1 - (s+t)^2}{1 - (s-t)^2}\right] \\ L_2(s,t) &= \frac{1}{2} \log \left[\frac{(1+t)^2 - s^2}{(1-t)^2 - s^2}\right] \end{aligned}\right\} \tag{33}$$

where s and t are dimensionless parameters given by

$$\left.\begin{aligned} s &= \frac{U_2}{U_1} \\ t &= \frac{U_3}{U_1} \end{aligned}\right\} \tag{34}$$

The relations between the Cartesian coordinates x_α and the local coordinates ξ_α involve the U's and V's of equations (31) and (32). They are

$$x_\alpha = V_{\alpha 1}\xi_1\xi_2 + V_{\alpha 2}\xi_1 + V_{\alpha 3}\xi_2 + \frac{x_\alpha^1 + x_\alpha^2 + x_\alpha^3 + x_\alpha^4}{4} \tag{35}$$

and

$$\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)} = U_1 - U_2\xi_2 - U_3\xi_1 \tag{36}$$

The numerical evaluation of the A-integrals and B-integrals by using equations (28) and (29) is a straightforward task. However, since the $\mathcal{F}$'s in equation (30) typically become infinite in the limit as s and/or t goes to zero, and since elements with small or vanishing s or t must often be evaluated, equation (30) is not a satisfactory form to use for the numerical evaluation of the C-integrals.

For the C-integrals used in this study, the difficulties are overcome by casting equation (30) in the form

$$C^{ij}_{\alpha\beta} = \frac{1}{U_1}\left\{ \bar{\mathcal{F}}^{ij}_{\alpha\beta 0} + \left[\bar{\mathcal{F}}^{ij}_{\alpha\beta 1} \quad \bar{\mathcal{F}}^{ij}_{\alpha\beta 2} \quad \bar{\mathcal{F}}^{ij}_{\alpha\beta 3}\right] \begin{bmatrix} \bar{L}_1(s,t) \\ \bar{L}_2(s,t) \\ \bar{L}_2(t,s) \end{bmatrix} \right\} \tag{37}$$

where

$$\left.\begin{aligned} \bar{L}_1(s,t) = \bar{L}_1(t,s) &= (\tilde{s}\tilde{t})^{-5}\left[L_1(s,t) + 2\tilde{s}\tilde{t} + \left(\frac{14}{3} - 2s^2 - 2t^2\right)(\tilde{s}\tilde{t})^3\right] \\ \bar{L}_2(s,t) &= (\tilde{t})^{-5}\left[L_2(s,t) - 2\tilde{t} - \left(\frac{2}{3} + 2s^2\right)\tilde{t}^3\right] \end{aligned}\right\} \tag{38}$$

and

$$\tilde{s} = \frac{s}{1 - t^2} \qquad \tilde{t} = \frac{t}{1 - s^2} \tag{39}$$

Since only convex quadrilaterals (those for which the Jacobian (36) does not vanish anywhere within the domain of the elements) are of interest, equation (37) needs to be evaluated only for r and s satisfying the condition

$$|s| + |t| < 1 \tag{40}$$

(Elements with $|s| + |t| = 1$ have three of their four vertexes lying on a straight line; that is, they have degenerated into triangles.) The $\overline{\mathcal{F}}$'s are free of singularities except at $s = \pm 1$ and $t = \pm 1$, and their numerical evaluation is straightforward. Elements with s or t equal or nearly equal to one are not of interest anyway.

Even though the functions L_1 and L_2 are free of singularities within the region specified by equation (40), they must be evaluated with care. Equation (38) is used directly for numerical evaluation only if neither s nor t is small; otherwise, Taylor series expansions are used. These series expansions were derived by using MACSYMA and are part of the computer program listed in reference 11.

Although the A-, B-, and C-integrals for any quadrilateral element can be numerically evaluated by using equations (28), (29), and (37), simplified forms for these integrals should be used to improve computational efficiency whenever the element is a trapezoid or a parallelogram.

Trapezoidal Elements

Trapezoidal elements are characterized by the vanishing of $t = U_3/U_1$ or else $s = U_2/U_1$. In the case $t = 0$, equation (37) reduces to

$$C^{ij}_{\alpha\beta} = \frac{1}{U_1}\left[\overline{g}^{ij}_{\alpha\beta 0} + \overline{g}^{ij}_{\alpha\beta 1}\,\overline{L}(s)\right] \tag{41}$$

where the $\overline{g}$'s are functions of s and the V's, and

$$\overline{L}(s) = s^{-5}\left[\log\left(\frac{1+s}{1-s}\right) - 2s - \frac{2}{3}s^3\right] \tag{42}$$

Similar expressions hold for the case $s = 0$. A Taylor series expansion is used to evaluate the function $\overline{L}$ when its argument is small.

Parallelogram Elements

Parellelogram elements (and thus rectangular elements) are characterized by the vanishing of both s and t. The formulas for the A-, B-, and C-integrals over quadrilateral elements (eqs. (28) to (30)) reduce in the parallelogram case to

$$A^{ijk\ell} = R_1^{ijk\ell}\,U_1 \tag{43}$$

$$B_{\alpha}^{ijk} = \begin{bmatrix} S_2^{ijk} & S_3^{ijk} \end{bmatrix} \begin{bmatrix} V_{\alpha 2} \\ V_{\alpha 3} \end{bmatrix} \tag{44}$$

$$C_{\alpha\beta}^{ij} = \frac{1}{U_1} \begin{bmatrix} V_{\alpha 2} & V_{\alpha 3} \end{bmatrix} \begin{bmatrix} T_1^{ij} & T_2^{ij} \\ T_3^{ij} & T_4^{ij} \end{bmatrix} \begin{bmatrix} V_{\beta 2} \\ V_{\beta 3} \end{bmatrix} \tag{45}$$

where the R's, S's, and T's are rational numbers. These expressions, like those for the triangular elements (eqs. (23) to (25)), are entirely free of logarithms.

GROUP-THEORETIC TECHNIQUES

The evaluation of the A-, B-, and C-integrals can be divided into two distinct phases: (1) obtaining analytic (or algebraic) expressions for the integrals using the symbolic-and-algebraic-manipulation language MACSYMA and (2) numerically evaluating the resulting expressions. These two phases will henceforth be referred to as the symbolic and numerical phases, respectively.

Because of the way in which the integrals B_{α}^{ijk} depend on the values of the Greek index α (see eq. (20)), separate symbolic integrations for fixed i,j,k are not required for B_1^{ijk} and B_2^{ijk}. Similarly, for given i,j only one symbolic integration is required for the set of integrals C_{11}^{ij}, C_{12}^{ij}, C_{21}^{ij}, and C_{22}^{ij}. Thus, by this consideration and by taking into account the symmetries of the Latin indices in equations (19), (20), and (21), the number of symbolic integrations required to determine the coefficients in the expressions (eqs. (23) to (25) and eqs. (28), (29), (37), (41), and (45)) for the A-, B-, and C-integrals would be $(r+4)!/(4!r!)$, $\frac{1}{2}(r+1)(r+2)(r)$, and $\frac{1}{2}(r)(r+1)$, respectively. (The $\overline{g}$'s in eq. (41) and the T's in eq. (45) do not require separate symbolic integrations since they can be determined from the $\overline{\mathcal{J}}$'s of eq. (37)). Thus, for the six elements developed in this study, a total of 4647 symbolic integrations would be required. The evaluation of such a large number of integrals symbolically, some of which are very complicated, seems discouraging even with the aid of a computer, and it is highly desirable to reduce this number. This reduction can be accomplished for the triangular and quadrilateral elements studied herein by taking advantage of the similarities in algebraic form among the various integrals, viz, by employing the theory of group transformations (refs. 9 and 10).

Basic Idea of Group-Theoretic Techniques

The essential idea of the group-theoretic techniques employed herein is that a relatively small number of the A-, B-, and C-integrals are selected as "representative" integrals for which symbolic calculations are performed directly. Each of the other A-, B-, and C-integrals is related to one of these representative integrals by symmetry considerations. Therefore, the symbolic phase of computation reduces to obtaining analytic expressions for just the representative integrals. The numbers of representative integrals required for each of the various elements considered in the present study is given in table II. The total number of representative integrals is 886; thus, the group-theoretic techniques reduce the number of symbolic calculations to less than one-fifth of what it would otherwise have been.

In this study, group transformations are applied by using two different (but mathematically equivalent) procedures. The first procedure is used for the sets of integrals that can be expressed as linear combinations (with numerical coefficients) of a small number of functions of the nodal coordinates. In this case the output from the symbolic phase of computation consists of arrays of integers obtained from the symbolic integration of representative integrals. In the numerical phase the coefficients for all the integrals related to these representative integrals are numerically evaluated and stored for repeated subsequent use. As a result, the evaluation of these integrals is very fast and only a relatively small amount of data needs to be transferred from the symbolic phase of computation to the numerical phase. This procedure was found to be the more efficient one for evaluating the A- and B-integrals (for all elements studied herein) and the C-integrals for the triangular and parallelogram elements. However, it would be very cumbersome to try to apply this procedure to the remaining C-integrals.

For evaluating the C-integrals for nonparallelogram quadrilateral elements, the second procedure involving group transformations was used. Since these integrals have more complicated forms, the output from the symbolic phase of computation consisted of algebraic expressions for the representative integrals rather than just arrays of coefficients. These algebraic expressions are used for the numerical evaluation of the nonrepresentative as well as of the representative C-integrals. This is done by applying suitable group transformations to the variables contained in these expressions before evaluating them. By using the same algebraic expressions for several related integrals, fewer symbolic calculations are performed and computer memory requirements for the numerical phase of computation are reduced.

The second procedure could have been applied to all the integrals but with some loss in the computational efficiency. The difference in the way the two procedures would apply to the A- and B-integrals is that by using the first procedure, the coefficients (the R's and S's) are transformed (and this needs to be done only once), and by using the second

procedure the functions of the nodal coordinates (the U's and V's) would be transformed (and this would have to be done for each element).

Before discussing the application of these two procedures in detail, the concept of dihedral groups is briefly reviewed.

Dihedral Groups

For triangular and quadrilateral elements the symmetry groups which serve to relate the various A-, B-, and C-integrals to their representative integrals belong to an infinite family of groups called dihedral groups. (See ref. 10.) For triangular elements the relevant group is the six-element dihedral group (denoted by $\mathcal{D}_3$), and for quadrilateral elements it is the eight-element dihedral group (denoted by $\mathcal{D}_4$). The group $\mathcal{D}_3$ is the symmetry group of an equilateral triangle, and the group $\mathcal{D}_4$ is the symmetry group of a square. These groups are relevant because any triangular element can appear as an equilateral triangle when viewed in an appropriate local coordinate system, and any quadrilateral element appears as a square as viewed in the quadrilateral coordinate system ξ_α of figure 3(a).

To each element in the symmetry group there corresponds a linear transformation on the local coordinate system which maps the element domain onto itself. Let such a transformation be symbolized by

$$\xi_\alpha \rightarrow \xi'_\alpha = \xi'_\alpha(\xi_\beta) \tag{46}$$

This transformation induces a transformation on any function $f(\xi_1,\xi_2)$ in the sense that

$$f(\xi_1,\xi_2) \rightarrow f'(\xi_1,\xi_2) \tag{47}$$

where

$$f'(\xi_1,\xi_2) = f(\xi'_1,\xi'_2) \tag{48}$$

In particular, the set of shape functions $N^i = N^i(\xi_1,\xi_2)$, $i = 1 \rightarrow r$, has the property that each transformed shape function $(N^i)'$ is identical to one of the untransformed functions N^j. In fact

$$(N^i)' = N^{\tau(i)} \tag{49}$$

where τ is a permutation on the set of r indices. Since the indices on the shape functions, in effect, form the superscripts on the A's, B's, and C's, the effect of the group transformations is to induce transformations on these superscripts.

To each element in the symmetry group, there also corresponds a permutation σ on the coordinates of the corner nodes

$$x_\alpha^i \rightarrow x_\alpha^{\sigma(i)} \qquad \begin{pmatrix} i = 3 & \text{for triangular elements;} \\ i = 4 & \text{quadrilateral elements} \end{pmatrix} \tag{50}$$

These permutations induce linear transformations on various sets of functions of the x^i. Examples of such sets are U_1; U_2 and U_3; $V_{\alpha 1}$; $V_{\alpha 2}$ and $V_{\alpha 3}$; r and s; L_1; L_2 and L_3; $\overline{L}_1$; $\overline{L}_2$ and $\overline{L}_3$; etc. Thus, for example,

$$U_1\left(x_\alpha^i\right) \rightarrow U_1\left(x_\alpha^{\sigma(i)}\right) = aU_1\left(x_\alpha^i\right) \tag{51}$$

and

$$\begin{bmatrix} U_2\left(x_\alpha^i\right) \\ U_3\left(x_\alpha^i\right) \end{bmatrix} \rightarrow \begin{bmatrix} U_2\left(x_\alpha^{\sigma(i)}\right) \\ U_3\left(x_\alpha^{\sigma(i)}\right) \end{bmatrix} = d \begin{bmatrix} U_2\left(x_\alpha^i\right) \\ U_3\left(x_\alpha^i\right) \end{bmatrix} \tag{52}$$

where a is +1 or -1 and d is a 2 by 2 matrix. Both a and d depend on the particular group transformation. In this way certain transformations on the superscripts of the $A^{ijk\ell}$, B_α^{ijk}, and $C_{\alpha\beta}^{ij}$ are related to linear transformations on the variables contained in the formulas for these integrals.

First Procedure Based on Group Transformations

As mentioned previously, group transformations are employed in two different ways. The first procedure based on group transformations is used to evaluate the R's, S's, and T's in equations (23) to (25), (28) to (29), and (43) to (45).

For triangular elements the formulas used to evaluate these coefficients are

$$R_1^{ijk\ell} = \left(\mathcal{R}_2^m\right)^{-1} \mathcal{R}_1^m \tag{53}$$

$$\left[S_1^{ijk} \quad S_2^{ijk}\right] = \left(\mathcal{S}_3^{\overline{m}}\right)^{-1}\left[\mathcal{S}_1^{\overline{m}} \quad \mathcal{S}_2^{\overline{m}}\right] d^{\overline{n}} \tag{54}$$

and

$$\begin{bmatrix} T_1^{ij} & T_2^{ij} \\ T_3^{ij} & T_4^{ij} \end{bmatrix} = \left(\mathcal{T}_5^{\overline{\overline{m}}}\right)^{-1}\left(d^{\overline{\overline{n}}}\right)^T \begin{bmatrix} \mathcal{T}_1^{\overline{\overline{m}}} & \mathcal{T}_2^{\overline{\overline{m}}} \\ \mathcal{T}_3^{\overline{\overline{m}}} & \mathcal{T}_4^{\overline{\overline{m}}} \end{bmatrix} d^{\overline{\overline{n}}} \tag{55}$$

where the d's are matrices which represent group transformations. The d's are given by

$$\left.\begin{aligned} d^1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad d^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad d^3 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\ d^4 &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad d^5 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad d^6 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}\right\} \tag{56}$$

In equations (53) to (55) T denotes transpose, and the $\mathcal{R}$'s, $\mathcal{S}$'s, and $\mathcal{T}$'s are integers obtained from the symbolic integration of representative integrals.

For quadrilateral elements the formulas used to evaluate the R's, S's, and T's are

$$R_1^{ijk\ell} = \left(\mathcal{R}_4^m\right)^{-1}\mathcal{R}_1^m$$

$$\left[R_2^{ijk\ell} \quad R_3^{ijk\ell}\right] = \left(\mathcal{R}_4^m\right)^{-1}\left[\mathcal{R}_2^m \quad \mathcal{R}_3^m\right] d^n \tag{57}$$

$$S_1^{ijk} = a^{\overline{n}}\left(\mathcal{S}_4^{\overline{m}}\right)^{-1}\mathcal{S}_1^{\overline{m}}$$

$$\left[S_2^{ijk} \quad S_3^{ijk}\right] = a^{\overline{n}}\left(\mathcal{S}_4^{\overline{m}}\right)^{-1}\left[\mathcal{S}_2^{\overline{m}} \quad \mathcal{S}_3^{\overline{m}}\right] d^{\overline{n}} \tag{58}$$

$$\begin{bmatrix} T_1^{ij} & T_2^{ij} \\ T_3^{ij} & T_4^{ij} \end{bmatrix} = \left(\mathcal{J}_5^{\bar{\bar{m}}}\right)^{-1} \left(d^{\bar{\bar{n}}}\right)^T \begin{bmatrix} \mathcal{J}_1^{\bar{\bar{m}}} & \mathcal{J}_2^{\bar{\bar{m}}} \\ \mathcal{J}_3^{\bar{\bar{m}}} & \mathcal{J}_4^{\bar{\bar{m}}} \end{bmatrix} d^{\bar{\bar{n}}} \tag{59}$$

where the a's and d's represent group transformations and are given by

$$a^1 = -a^2 = a^3 = -a^4 = a^5 = -a^6 = a^7 = -a^8 = 1 \tag{60}$$

$$\left.\begin{aligned} d^1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad d^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad d^3 = -d^1 \qquad d^4 = -d^2 \\ d^5 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad d^6 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad d^7 = -d^5 \qquad d^8 = -d^6 \end{aligned}\right\} \tag{61}$$

and the $\mathcal{R}$'s, $\mathcal{S}$'s, and $\mathcal{J}$'s are again integers obtained from the symbolic integration of representative integrals.

In equations (53) to (55) and (57) to (59) the superscripts m, $\bar{m}$, $\bar{\bar{m}}$, n, $\bar{n}$, and $\bar{\bar{n}}$ in the right-hand members must be understood as integer-valued functions of the superscripts in the corresponding left-hand members, that is

$$\left.\begin{aligned} m &= m(i,j,k,\ell) \qquad \bar{m} = \bar{m}(i,j,k) \qquad \bar{\bar{m}} = \bar{\bar{m}}(i,j) \\ n &= n(i,j,k,\ell) \qquad \bar{n} = \bar{n}(i,j,k) \qquad \bar{\bar{n}} = \bar{\bar{n}}(i,j) \end{aligned}\right\} \tag{62}$$

The superscripts m, $\bar{m}$, and $\bar{\bar{m}}$ indicate which representative integral is being referenced, and the superscripts n, $\bar{n}$, and $\bar{\bar{n}}$ indicate which group transformation is being applied. The dependence of the superscripts m, $\bar{m}$, $\bar{\bar{m}}$, n, $\bar{n}$, and $\bar{\bar{n}}$ on the superscripts i, j, k, and ℓ is given by a set of integer-valued input arrays generated by separate computer programs. These arrays are given in reference 11.

Whenever the left-hand member of one of the equations (53) to (55) or (57) to (59) is one of the representative integrals, then the superscript n or $\bar{n}$ or $\bar{\bar{n}}$ (as the case may be) equals one which indicates the identity group transformation. Thus for triangular elements the coefficients for the representative A-, B-, and C-integrals are $\mathcal{R}_1^m / \mathcal{R}_2^m$;

$\mathcal{S}_1^{\overline{m}}/\mathcal{S}_3^{\overline{m}}$ and $\mathcal{S}_2^{\overline{m}}/\mathcal{S}_3^{\overline{m}}$; and $\mathcal{T}_1^{\overline{\overline{m}}}/\mathcal{T}_5^{\overline{\overline{m}}}$, $\mathcal{T}_2^{\overline{\overline{m}}}/\mathcal{T}_5^{\overline{\overline{m}}}$, $\mathcal{T}_3^{\overline{\overline{m}}}/\mathcal{T}_5^{\overline{\overline{m}}}$, and $\mathcal{T}_4^{\overline{\overline{m}}}/\mathcal{T}_5^{\overline{\overline{m}}}$; respectively; and similarly for quadrilateral elements. The arrays of $\mathcal{R}$-, $\mathcal{S}$-, and $\mathcal{T}$-values are given in reference 11.

First Procedure Illustrated

To illustrate the use of this procedure, consider the integral A^{2111} for the ST4 element. Since A^{2111} was selected as one of the representative integrals, it has $n = 1$. From the way in which the representative integrals have been ordered, A^{2111} is the eighth representative A-integral, and thus it has $m = 8$. Because A^{2111} is a representative integral, it is one of the integrals to be symbolically integrated. The result of this integration is

$$A^{2111} = \frac{-U_1 - 2U_2 + 3U_3}{75} \tag{63}$$

and thus, by comparing equations (57) and (61) with equation (27), it follows that

$$\left.\begin{aligned} \mathcal{R}_1^8 &= -1 & \mathcal{R}_2^8 &= -2 \\ \mathcal{R}_3^8 &= 3 & \mathcal{R}_4^8 &= 75 \end{aligned}\right\} \tag{64}$$

These four values are included in the $\mathcal{R}$-array for the ST4 element assembled during the symbolic phase of the computation.

Seven other A-integrals for the ST4 element are also characterized by $m = 8$. They are A^{2221}, A^{3222}, A^{3332}, A^{4333}, A^{4443}, A^{4441}, and A^{4111}, and their n values are 5, 4, 6, 2, 8, 3, and 7, respectively. Thus, from equations (52) and (56), it follows that their coefficients are

$$\left.\begin{aligned} R_1^{2111} &= -0.0133 & R_2^{2111} &= -0.0267 & R_3^{2111} &= 0.0400 \\ R_1^{2221} &= -0.0133 & R_2^{2221} &= -0.0267 & R_3^{2221} &= -0.0400 \\ R_1^{3222} &= -0.0133 & R_2^{3222} &= -0.0400 & R_3^{3222} &= -0.0267 \\ R_1^{3332} &= -0.0133 & R_2^{3332} &= -0.0400 & R_3^{3332} &= 0.0267 \end{aligned}\right\} \tag{65}$$

(Equations continued on next page)

$$\left.\begin{array}{lll} R_1^{4333} = -0.0133 & R_2^{4333} = 0.0400 & R_3^{4333} = 0.0267 \\ R_1^{4443} = -0.0133 & R_2^{4443} = 0.0400 & R_3^{4443} = -0.0267 \\ R_1^{4441} = -0.0133 & R_2^{4441} = 0.0267 & R_3^{4441} = -0.0400 \\ R_1^{4111} = -0.0133 & R_2^{4111} = 0.0267 & R_3^{4111} = 0.0400 \end{array}\right\} \quad (65)$$

These results are included in the R-array which is evaluated in the numerical phase of computation and stored for subsequent use. Then for each finite element processed, the U's are evaluated by using equation (30) and after this evaluation the set of distinct A-integrals (that is, the $A^{ijk\ell}$ with $i \geq j \geq k \geq \ell$) is numerically evaluated and saved in an array. Included in this A-array are the eight integrals A^{2111}, A^{2221}, A^{3222}, A^{3332}, A^{4333}, A^{4443}, A^{4441}, and A^{4111}, whose coefficients are given in equations (65). Finally, as various A-integrals are needed in the evaluation of the element characteristic matrices, they are quickly retrieved from the A-array. Because of the permutational symmetry of the indices, the 32 A-integrals on the following list all depend on the $\mathcal{R}$-values given in equations (64):

$$A^{2111} = A^{1211} = A^{1121} = A^{1112} \qquad A^{1222} = A^{2122} = A^{2212} = A^{2221}$$

$$A^{3222} = A^{2322} = A^{2232} = A^{2223} \qquad A^{2333} = A^{3233} = A^{3323} = A^{3332}$$

$$A^{4333} = A^{3433} = A^{3343} = A^{3334} \qquad A^{3444} = A^{4333} = A^{4434} = A^{4443}$$

$$A^{1444} = A^{4144} = A^{4414} = A^{4441} \qquad A^{4111} = A^{1411} = A^{1141} = A^{1114}$$

In this way there can be as many as 192 A-integrals associated with a single representative A-integral.

Second Procedure Based on Group Transformations

The second procedure based on group transformations is used for the evaluation of C-integrals for nonparallelogram quadrilateral elements. Because of the greater complexity of the expressions for these integrals, the output from the symbolic phase of computation consists of FORTRAN-coded formulas for the representative integrals rather than arrays of integers as in the other cases. These formulas are functions of the variables U_1, s, t, $\overline{L}_1(s,t)$, $\overline{L}_2(s,t)$, $\overline{L}_2(t,s)$, and the V's. The procedure by which both a representative C-integral and the integrals related to it by dihedral symmetry can be evaluated by use of many of the same FORTRAN statements is the following:

(1) Based on the superscripts i and j, the program looks up the values of $\overline{\overline{m}} = \overline{\overline{m}}(i,j)$ and $\overline{\overline{n}} = \overline{\overline{n}}(i,j)$, where $\overline{\overline{m}}$ designates the representative integrals and $\overline{\overline{n}}$ designates the group transformation.

(2) The quantities $V_{\alpha 1}$, $V_{\alpha 2}$, $V_{\alpha 3}$, s, t, $\overline{L}_2(s,t)$, and $\overline{L}_2(t,s)$ undergo the following transformations:

$$\left.\begin{aligned} V_{\alpha 1} \rightarrow V_{\alpha 1}^{\overline{\overline{n}}} &= a^{\overline{\overline{n}}} V_{\alpha 1} \\ \begin{bmatrix} V_{\alpha 2} \\ V_{\alpha 3} \end{bmatrix} \rightarrow \begin{bmatrix} V_{\alpha 2}^{\overline{\overline{n}}} \\ V_{\alpha 3}^{\overline{\overline{n}}} \end{bmatrix} &= a^{\overline{\overline{n}}}\, d^{\overline{\overline{n}}} \begin{bmatrix} V_{\alpha 2} \\ V_{\alpha 3} \end{bmatrix} \end{aligned}\right\} \tag{66}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} \rightarrow \begin{bmatrix} s^{\overline{\overline{n}}} \\ t^{\overline{\overline{n}}} \end{bmatrix} = d^{\overline{\overline{n}}} \begin{bmatrix} s \\ t \end{bmatrix} \tag{67}$$

$$\begin{bmatrix} \overline{L}_2(s,t) \\ \overline{L}_2(t,s) \end{bmatrix} \rightarrow \begin{bmatrix} \overline{L}_2^{\overline{\overline{n}}}(s,t) \\ \overline{L}_2^{\overline{\overline{n}}}(t,s) \end{bmatrix} = e^{\overline{\overline{n}}} \begin{bmatrix} \overline{L}_2(s,t) \\ \overline{L}_2(t,s) \end{bmatrix} \tag{68}$$

where

$$\left.\begin{aligned} e^1 = e^3 = e^5 = e^7 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ e^2 = e^4 = e^6 = e^8 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}\right\} \tag{69}$$

and the a's and d's are given in equations (60) and (61). (Computationally, it is much faster to use eqs. (65) than to reevaluate the logarithmic functions each time s and t are transformed.) The variables U_1 and $\overline{L}_1^{\overline{\overline{n}}}(s,t) = \overline{L}_1(s^{\overline{\overline{n}}}, t^{\overline{\overline{n}}})$ are not transformed.

(3) The FORTRAN code for the $\overline{\overline{m}}$th representative integral is executed by using these transformed variables in place of the original variables. This technique of using the same FORTRAN statements to evaluate several symmetry-related integrals significantly reduces the program length.

PERFORMANCE EVALUATION AND COMPARISON WITH NUMERICAL QUADRATURE

In order to assess the computational efficiency of the symbolic integration approach for developing elemental matrices, a computer program called SYMINSE (SYMbolically INtegrated Shell Elements) was developed based on this approach. The program generates the element characteristic arrays (stiffness, geometric stiffness, mass, and load matrices) for the six elements in table I. A detailed description of SYMINSE program is given in reference 11.

The CPU times required for the SYMINSE program to generate the element characteristic arrays for 14 different shallow-shell elements and 14 flat-plate elements (with bending extensional coupling included) are given in table III. Also included in this table are the field lengths required for these elements. No comparison is given of the CPU times required for computing the characteristic arrays for these elements via the symbolic integration approach compared with the numerical quadrature approach. This is because any such comparison would, by necessity, be influenced by the efficiencies of both programs and would be dependent on compilers, operating systems, and computer hardware as well.

Instead a quantitative measure of the efficiency of the symbolic integration approach is provided by giving a comparison of the number of floating-point multiplications and additions in a conventional numerical quadrature approach with the number required by the SYMINSE program.

In the conventional numerical quadrature approach, the element stiffness matrix is expressed in the form

$$[K] = \sum_{\ell=1}^{\mu} H_\ell [B_\ell]^T_{5r,8} [C]_{8,8} [B_\ell]_{8,5r} \tag{70}$$

where $[B_\ell]$ is the strain-displacement matrix for interpolation models, $[C]$ is the matrix of shell stiffnesses, H_ℓ are the weighting coefficients for numerical quadrature, μ is the number of quadrature points, and r is the number of shape functions.

To generate the element stiffness matrix [K], the matrix of shell stiffnesses [C] is multiplied in turn by each of the weight coefficients H_ℓ. Because of the symmetry of the element stiffness matrix, only the upper triangular part of [K] needs to be evaluated. The individual contributions at each of the quadrature points are added to give the stiffness matrix.

Table IV gives the number of primary floating-point arithmetic operations required to generate the matrix [K] as a function of the number of numerical quadrature points μ and the number of shape functions r. The numbers given in this table are based on the assumption that each element of $[B_\ell]$ is a single quantity (rather than an expression in the local coordinates) and therefore, the operation counts given are conservative, that is, less than the actual number of arithmetic operations.

The number of floating-point arithmetic operations required to generate the element characteristic arrays for triangular and parallelogram elements using the SYMINSE program are given in table V. Note that very few additional operations are required for the evaluation of the geometric stiffness, the consistent mass, and the consistent load coefficients. The trapezoidal and nontrapezoidal quadrilateral (trapezium) elements are not included in table V because their arithmetic operation counts are more difficult to obtain. However, the CPU times listed in table III give a measure of the additional complexity for these cases.

As a test for the reliability of the operation count estimates, the ratios of the CPU times expended in evaluating integrals compared with the CPU times expended in forming the characteristic arrays as linear combinations of integrals are given in table VI. These ratios may be compared with the comparable ratios of operation counts also given in table VI. The ratios are not very close but agree in order of magnitude.

Figure 4 shows the ratios of the arithmetic operation counts involved in the evaluation of the element stiffness array for both the symbolic manipulation and numerical quadrature procedures.

The element characteristic arrays computed by the SYMINSE program have been verified by comparing them with those obtained by a program based on numerical quadrature. This latter program, although not optimized for efficiency, ran much slower than the SYMINSE program. This fact together with the preceding comparison of operation counts strongly suggest that the symbolic integration approach presented herein holds much promise.

POSSIBLE FURTHER IMPROVEMENTS AND EXTENSIONS

On the basis of the present study, the following remarks regarding the use of symbolic integration in the development of higher order elements seem to be in order:

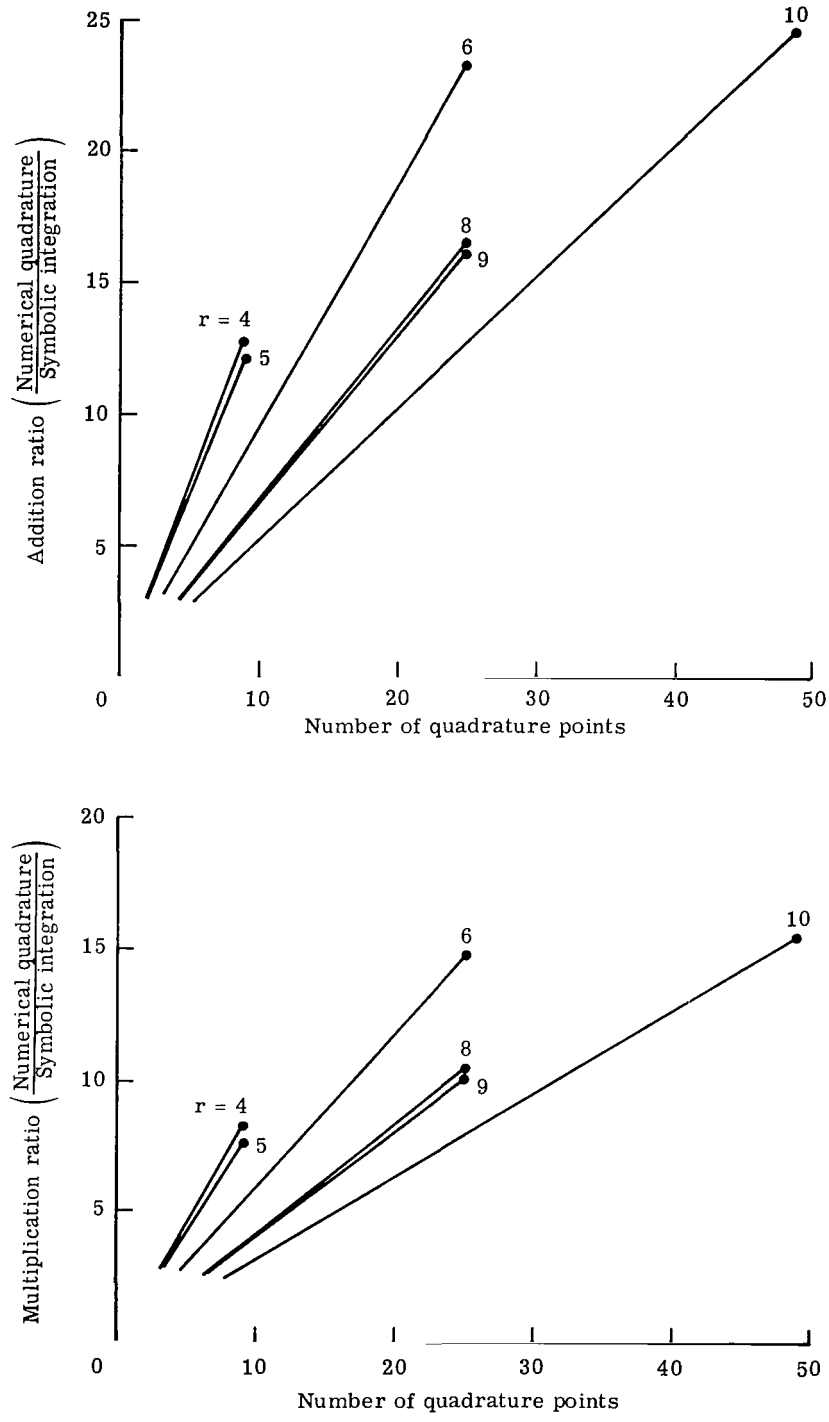


Figure 4.- Comparison of number of floating-point arithmetic operations required to evaluate integrals using numerical quadrature and symbolic integration techniques. r, number of shape functions.

(1) The analytic methods discussed herein can be easily implemented without modification for triangular elements with more than ten nodes and for parallelogram elements with more than eight nodes.

(2) A minor modification of these methods is required for trapezoidal and trapezium (nontrapezoid quadrilateral) elements with more than eight nodes. For these elements the $\mathcal{F}$'s and $\mathcal{G}$'s of equations (37) and (41) contain singularities when s or t equals zero. These difficulties are eliminated by using logarithmic functions obtained by subtracting off low order terms in the Taylor series expansions of $\overline{L}(s)$, $\overline{L}_1(s,t)$, and $\overline{L}_2(s,t)$ defined in equations (42) and (38).

(3) A minor modification of the way in which symmetry transformations are applied is required for quadrilateral elements with more than one bubble mode. This case is discussed in reference 9.

(4) With these modifications the implementation of trapezoidal elements with more than eight nodes and/or with more than one bubble mode is not difficult. However, the implementation of trapezium elements with more than eight nodes or with more than one bubble mode would be difficult because of the complexity of the analytic expressions.

(5) For the trapezium elements a hybrid approach which combines the exact symbolic integration with either approximate analytic integration or numerical quadrature appears to have advantages over the purely numerical quadrature and the purely symbolic integration approaches. In the hybrid approach the A- and B-integrals are evaluated exactly by using symbolic integration and the C-integrals are evaluated approximately by numerical quadrature or symbolic integration. This approach would retain the major advantages resulting from symbolic integration of the A- and B-integrals and eliminate the difficulties associated with exact symbolic integration of the C-integrals.

A count of floating-point arithmetic operations suggests that even for a purely numerical quadrature approach, the evaluation of A-, B-, and C-integrals followed by formation of the stiffness coefficients as a linear combination of these integrals is faster than the conventional approach discussed in the preceding section. This suggests that symbolic integration and numerical quadrature can be readily combined in one program.

CONCLUDING REMARKS

Computerized symbolic integration and group-theoretic techniques are employed in combination to obtain analytic expressions for the stiffness, geometric stiffness, consistent mass and consistent load coefficients of composite shallow-shell structural elements. The elements are shear flexible and have variable curvature. A stiffness (displacement) formulation is used with the fundamental unknowns consisting of both the displacement and rotation components of the reference surface of the shell. Both triangular and quadrilat-

eral elements are developed. The triangular elements developed have six and ten nodes. The quadrilateral elements have four and eight nodes and can have internal degrees of freedom associated with displacement modes which vanish along the edges of the element.

The coefficients of the elemental matrices (stiffness, geometric stiffness, consistent mass, and consistent load) are expressed as linear combinations of integrals (over the element domain) whose integrands are products of shape functions and their derivatives. The evaluation of the elemental matrices is thus divided into two separate problems – determination of the coefficients in the linear combination and evaluation of the integrals. The latter problem can be computationally the more time consuming and the present study aims at reducing the computational effort involved in this task through the use of computerized symbolic integration.

Based upon a comparison between the number of floating-point arithmetic operations required in the symbolic integration and traditional numerical quadrature approaches, it appears that the symbolic manipulation approach, as presented herein, is considerably more efficient and holds much promise. This is especially true for the triangular, parallelogram, and trapezoidal elements.

For trapezium (nontrapezoidal quadrilateral) elements, however, a hybrid approach in which some of the integrals (the A- and B-integrals) are evaluated exactly by symbolic manipulation and other integrals (the C-integrals) by either numerical quadrature or by means of approximate analytic expressions appears to have advantages over a purely numerical quadrature or a purely symbolic integration approach.

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October 3, 1975

APPENDIX A

ELASTIC COEFFICIENTS OF LAMINATED SHELLS

Elastic Stiffnesses of Layers

The nonzero stiffness coefficients $c^{(k)}_{\alpha\beta\gamma\rho}$ and $c^{(k)}_{\alpha 3\beta 3}$ of the kth orthotropic layer of the shell referred to the directions of principal elasticity are given by

$$c^{(k)}_{1111} = E^{(k)}_L \Big/ \bar{\lambda}^{(k)}$$

$$c^{(k)}_{1122} = \nu^{(k)}_{LT} E^{(k)}_T \Big/ \bar{\lambda}^{(k)}$$

$$c^{(k)}_{2222} = E^{(k)}_T \Big/ \bar{\lambda}^{(k)}$$

$$c^{(k)}_{1212} = G^{(k)}_{LT}$$

$$c^{(k)}_{1313} = G^{(k)}_{LT}$$

$$c^{(k)}_{2323} = G^{(k)}_{TT}$$

where the subscripts L and T denote the direction of fibers and the transverse direction, respectively, the superscript k refers to the kth layer, the E's are elastic moduli, the G's are shear moduli, and ν_{LT} is the Poisson's ratio measuring the strain in the T-direction due to a uniaxial normal stress in the L-direction

$$\nu_{TL} E_L = \nu_{LT} E_T$$

$$\bar{\lambda} = 1 - \nu_{LT}\nu_{TL}$$

The stiffness coefficients $c_{\alpha\beta\gamma\rho}$ and $c_{\alpha 3\beta 3}$ satisfy the following symmetry relationships:

$$c_{\alpha\beta\gamma\rho} = c_{\gamma\rho\alpha\beta} = c_{\beta\alpha\gamma\rho} = c_{\alpha\beta\rho\gamma}$$

and

$$c_{\alpha 3 \beta 3} = c_{\beta 3 \alpha 3} = c_{3 \alpha \beta 3} = c_{\alpha 33 \beta}$$

If the coordinates x_α are rotated, the elastic coefficients $c_{\alpha\beta\gamma\rho}$ and $c_{\alpha 3 \beta 3}$ transform as components of fourth- and second-order tensors, respectively. The transformation law of these coefficients is expressed as follows:

$$c_{\alpha'\beta'\gamma'\rho'} = c_{\alpha\beta\gamma\rho}\, \ell_{\alpha\alpha'}\, \ell_{\beta\beta'}\, \ell_{\gamma\gamma'}\, \ell_{\rho\rho'}$$

and

$$c_{\alpha' 3 \beta' 3} = c_{\alpha 3 \beta 3}\, \ell_{\alpha\alpha'}\, \ell_{\beta\beta'}$$

where $c_{\alpha'\beta'\gamma'\rho'}$ and $c_{\alpha' 3 \beta' 3}$ are the elastic coefficients referred to the new coordinate system x_α, and

$$\ell_{\alpha\alpha'} = \cos\,(x_\alpha, x_{\alpha'})$$

Elastic Coefficients of Shell

The equivalent elastic stiffnesses of the shell are given by

$$\left[C_{\alpha\beta\gamma\rho} \quad F_{\alpha\beta\gamma\rho} \quad D_{\alpha\beta\gamma\rho}\right] = \sum_{k=1}^{NL} \int_{h_{k-1}}^{h_k} c^{(k)}_{\alpha\beta\gamma\rho} \left[1 \quad x_3 \quad x_3^2\right] dx_3$$

and

$$C_{\alpha 3 \beta 3} = \sum_{k=1}^{NL} \int_{h_{k-1}}^{h_k} c^{(k)}_{\alpha 3 \beta 3}\, dx_3$$

where NL is the total number of layers of the shell and h_k and h_{k-1} are the distances from the reference surface to the top and bottom surfaces of the kth layer, respectively. The elastic compliances of the shell $A_{\alpha\beta\gamma\rho}$, $B_{\alpha\beta\gamma\rho}$, $G_{\alpha\beta\gamma\rho}$, and $A_{\alpha 3 \beta 3}$ are obtained by inversion of the matrix of the elastic stiffnesses. (See ref. 5.)

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The shell stiffnesses and compliance coefficients satisfy symmetry and transformation relations similar to those of the stiffness coefficients of individual layers.

The density parameters of the shell are given by

$$\begin{bmatrix} m_0 & m_1 & m_2 \end{bmatrix} = \sum_{k=1}^{NL} \int_{h_{k-1}}^{h_k} \rho_s^{(k)} \begin{bmatrix} 1 & x_3 & x_3^2 \end{bmatrix} dx_3$$

where $\rho_s^{(k)}$ is the mass density of the kth layer of the shell.

APPENDIX B

FORMULAS FOR THE COEFFICIENTS IN GOVERNING EQUATIONS FOR INDIVIDUAL ELEMENTS

The expressions for the independent stiffness coefficients in equation (12) are given by

$$K_{\alpha\beta}^{ij} = \int_{\Omega(e)} C_{\alpha\gamma\beta\rho}\ \partial_\gamma N^i\ \partial_\rho N^j\ d\Omega$$

$$K_{\alpha 3}^{ij} = \int_{\Omega(e)} C_{\alpha\beta\gamma\rho}\ k_{\beta\rho}^{\ell}\ \partial_\gamma N^i\ N^j N^\ell\ d\Omega$$

$$K_{\alpha,\beta+3}^{ij} = \int_{\Omega(e)} F_{\alpha\beta\gamma\rho}\ \partial_\gamma N^i\ \partial_\rho N^j\ d\Omega$$

$$K_{33}^{ij} = \int_{\Omega(e)} \left(C_{\alpha\gamma\beta\rho}\ k_{\alpha\gamma}^{\ell}\ k_{\beta\rho}^{m}\ N^i N^j N^\ell N^m + C_{\alpha 3\beta 3}\ \partial_\alpha N^i\ \partial_\beta N^j\right) d\Omega$$

$$K_{3,\alpha+3}^{ij} = \int_{\Omega(e)} \left(F_{\alpha\gamma\beta\rho}\ k_{\beta\rho}^{\ell}\ \partial_\gamma N^i\ N^j N^\ell + C_{\alpha 3\beta 3}\ N^i\ \partial_\beta N^j\right) d\Omega$$

$$K_{\alpha+3,\beta+3}^{ij} = \int_{\Omega(e)} \left(D_{\alpha\gamma\beta\rho}\ \partial_\gamma N^i\ \partial_\rho N^j + C_{\alpha 3\beta 3}\ N^i N^j\right) d\Omega$$

The independent nonzero geometric stiffness coefficients are given by

$$\overline{K}_{33}^{ij} = \int_{\Omega(e)} \mathcal{N}_{\alpha\beta}^{o}\ \partial_\alpha N^i\ \partial_\beta N^j\ d\Omega$$

The independent nonzero consistent mass coefficients are given by

$$M_{\alpha\beta}^{ij} = \int_{\Omega(e)} m_0\ \delta_{\alpha\beta}\ N^i N^j\ d\Omega$$

$$M_{\alpha,\beta+3}^{ij} = \int_{\Omega(e)} m_1\ \delta_{\alpha\beta}\ N^i N^j\ d\Omega$$

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$$M_{33}^{ij} = \int_{\Omega(e)} m_0 N^i N^j \, d\Omega$$

$$M_{\alpha+3,\beta+3}^{i\ \ j} = \int_{\Omega(e)} m_2 \, \delta_{\alpha\beta} \, N^i N^j \, d\Omega$$

where $\delta_{\alpha\beta}$ is the Kronecker delta on α and β.

Finally, the consistent nodal load coefficients are given by

$$P_\alpha^\ell = \int_{\Omega(e)} N^\ell N^m p_\alpha^m \, d\Omega$$

$$P_3^\ell = \int_{\Omega(e)} N^\ell N^m p^m \, d\Omega$$

In these equations the range of the lower case Latin indices is $1 \rightarrow r$ where r is the number of shape functions, the range of the Greek indices is 1,2, and a repeated index denotes summation over the full range of the index. However, for the elements with internal degrees of freedom (SQ5 and SQ9) the indices ℓ and m in these expressions have a range equal to the number of nodes in the element (that is, 4 for the SQ5 element and 8 for the SQ9 element). This amounts to distributing the loading on the nodes of these elements with no loading associated with the internal degrees of freedom.

APPENDIX C

SHAPE FUNCTIONS USED IN THE PRESENT STUDY

The shape functions used in the present study are numbered as shown in figure 2.

Four-Node Quadrilateral Elements (SQ4 and SQ5)

The shape functions for SQ4 are

$$N^1 = \frac{(1 - \xi_1)(1 - \xi_2)}{4}$$

$$N^2 = \frac{(1 + \xi_1)(1 - \xi_2)}{4}$$

$$N^3 = \frac{(1 + \xi_1)(1 + \xi_2)}{4}$$

$$N^4 = \frac{(1 - \xi_1)(1 + \xi_2)}{4}$$

The shape functions for SQ5 are these functions plus the additional function representing the bubble mode.

$$N^5 = \left(1 - \xi_1^2\right)\left(1 - \xi_2^2\right)$$

Eight-Node Quadrilateral Elements (SQ8 and SQ9)

The shape functions for SQ8 are

$$N^1 = \frac{(1 - \xi_1)(1 - \xi_2)(-1 - \xi_1 - \xi_2)}{4}$$

$$N^2 = \frac{(1 + \xi_1)(1 - \xi_2)(-1 + \xi_1 - \xi_2)}{4}$$

$$N^3 = \frac{(1 + \xi_1)(1 + \xi_2)(-1 + \xi_1 + \xi_2)}{4}$$

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$$N^4 = \frac{(1 - \xi_1)(1 + \xi_2)(-1 - \xi_1 + \xi_2)}{4}$$

$$N^5 = \frac{(1 - \xi_1^2)(1 - \xi_2)}{2}$$

$$N^6 = \frac{(1 - \xi_1)(1 - \xi_2^2)}{2}$$

$$N^7 = \frac{(1 - \xi_1^2)(1 + \xi_2)}{2}$$

$$N^8 = \frac{(1 + \xi_1)(1 - \xi_2^2)}{2}$$

For SQ9 the additional shape function N^9 is

$$N^9 = (1 - \xi_1^2)(1 - \xi_2^2)$$

Triangular Elements (ST6 and ST10)

The shape functions for ST6 are

$$N^1 = \xi_1(2\xi_1 - 1)$$

$$N^2 = \xi_2(2\xi_2 - 1)$$

$$N^3 = (1 - \xi_1 - \xi_2)(1 - 2\xi_1 - 2\xi_2)$$

$$N^4 = 4\xi_1\xi_2$$

$$N^5 = 4\xi_2(1 - \xi_1 - \xi_2)$$

$$N^6 = 4\xi_1(1 - \xi_1 - \xi_2)$$

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The shape functions for ST10 are

$$N^1 = \frac{\xi_1(3\xi_1 - 1)(3\xi_1 - 2)}{2}$$

$$N^2 = \frac{\xi_2(3\xi_2 - 1)(3\xi_2 - 2)}{2}$$

$$N^3 = \frac{(1 - \xi_1 - \xi_2)(2 - 3\xi_1 - 3\xi_2)(1 - 3\xi_1 - 3\xi_2)}{2}$$

$$N^4 = \frac{9\xi_1\xi_2(3\xi_1 - 1)}{2}$$

$$N^5 = \frac{9\xi_2(3\xi_2 - 1)(1 - \xi_1 - \xi_2)}{2}$$

$$N^6 = \frac{9\xi_1(1 - \xi_1 - \xi_2)(2 - 3\xi_1 - 3\xi_2)}{2}$$

$$N^7 = \frac{9\xi_1\xi_2(3\xi_2 - 1)}{2}$$

$$N^8 = \frac{9\xi_2(1 - \xi_1 - \xi_2)(2 - 3\xi_1 - 3\xi_2)}{2}$$

$$N^9 = \frac{9\xi_1(3\xi_1 - 1)(1 - \xi_1 - \xi_2)}{2}$$

$$N^{10} = 27\xi_1\xi_2(1 - \xi_1 - \xi_2)$$

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TABLE I.- CHARACTERISTICS OF THE STIFFNESS FINITE-ELEMENT MODELS DEVELOPED IN THE PRESENT STUDY

Element shape	Number of nodes per element	Number of internal degrees of freedom per unknown	Approximation	Total number of degrees of freedom, 5r	Designation
Quadrilateral	4	None	Bilinear	20	SQ4
		1	Bilinear plus bubble function	25	SQ5
	8	None	Quadratic	40	SQ8
		1	Quadratic plus bubble function	45	SQ9
Triangular	6	None	Quadratic	30	ST6
	10	None	Cubic	50	ST10

TABLE II.- THE NUMBER OF SYMBOLIC INTEGRATIONS NEEDED FOR TRIANGULAR AND QUADRILATERAL ELEMENTS

Element shape	Number of shape functions per element, r	$A^{ijk\ell}$ (i, j, k, ℓ = 0 – r)		B_{α}^{ijk} (i, j = 0 – r; k = 1 – r; α = 1,2)		$C_{\alpha\beta}^{ij}$ (i, j = 1 – r; α, β = 1,2)	
		Number of numerically distinct integrals, $(r+4)!/(4!r!)$	Number of representative integrals required	Number of numerically distinct integrals $r(r+1)(r+2)$	Number of representative integrals required	Number of numerically distinct integrals, $r(2r+1)$	Number of representative integrals required
Triangular	6	210	51	336	36	78	6
	10	1001	195	1320	121	210	13
Quadrilateral	4	70	17	120	11	36	3
	5	126	34	210	24	55	5
	8	495	84	720	54	136	8
	9	715	130	990	83	171	11

TABLE III. - CPU TIMES AND FIELD LENGTHS ON A CDC 6600 COMPUTER (USING THE RUN COMPILER UNDER SCOPE 3.0 OPERATING SYSTEM) FOR 14 SAMPLE PROBLEMS

Element designation	Element shape	Required field lengths* (octal)	CPU time (in milliseconds) to compute A-, B-, and C-integrals	Total CPU time (in milliseconds), plate elements**	Total CPU time (in milliseconds), shell elements
SQ4	Parallelogram	33 456	8	22	36
	Trapezoid		12	26	40
	Trapezium		18	32	46
SQ5	Parallelogram	34 276	10	30	52
	Trapezoid		16	36	58
	Trapezium		32	52	74
ST6	Triangle	35 270	12	40	96
SQ8	Parallelogram	40 167	18	62	212
	Trapezoid	40 167	44	88	238
	Trapezium	42 662	170	214	364
SQ9	Parallelogram	42 077	20	76	260
	Trapezoid	42 077	58	114	298
	Trapezium	44 572	212	268	452
ST10	Triangle	44 257	28	98	414

*Initially 6000 octal words more are needed to accommodate the loader and load tables.

**Includes bending-extensional coupling.

TABLE IV.- AN ESTIMATE FOR THE NUMBER OF ARITHMETIC OPERATIONS REQUIRED TO GENERATE THE ELEMENT STIFFNESS MATRIX USING THE CONVENTIONAL NUMERICAL QUADRATURE PROCEDURE

$$\left[[K] = \sum_{\ell=1}^{\mu} H_\ell \, [B_\ell]^T_{5r,8} \, [C]_{8,8} \, [B_\ell]_{8,5r} \right]$$

Computation step	Operations	
	Additions	Multiplications
Product of weighting coefficients and matrix $[C]$, $[\overline{C}]_{8,8} = H_\ell \, [C]_{8,8}$	0	$(8)(8)\mu$
Product of $[\overline{C}_\ell]_{8,8} \, [B_\ell]_{8,5r}$	$(8)(8)(5r)\mu$	$(8)(8)(5r)\mu$
Product of $[B_\ell]^T_{5r,8} \left([\overline{C}_\ell] \, [B_\ell] \right)_{8,5r}$	$\frac{(5r)(5r+1)}{2}(8)\mu$	$\frac{(5r)(5r+1)}{2}(8)\mu$
Summation over μ quadrature points	$\frac{(5r)(5r+1)}{2}\mu$	0
Total	$\frac{5}{2} r\mu(45r + 137)$	$(100r^2 + 340r + 64)\mu$

TABLE V. - NUMBER OF FLOATING-POINT OPERATIONS REQUIRED TO GENERATE THE ELEMENT CHARACTERISTIC ARRAYS USING THE SYMINSE PROGRAM

Computation step	Operations	Triangular elements		Parallelogram (or rectangular) elements			
		ST6	ST10	SQ4	SQ5	SQ8	SQ9
Evaluations of integrals	Additions	515	1 771	221	351	1 029	1 371
	Multiplications	1306	4 745	524	860	2 669	3 609
Formation of linear combinations for stiffness K_{IJ}^{ij}	Additions	6132	27 830	2030	3045	14 148	17 685
	Multiplications	8490	38 510	2804	4204	19 592	24 488
Formation of linear combinations for arrays $\bar{K}_{IJ}^{ij}$, P_J^j, and $\mathcal{M}_{IJ}^{ij}$	Additions	99	265	46	61	172	199
	Multiplications	136	366	63	87	237	281
Sum of these three steps	Additions	6746	29 866	2297	3457	15 349	19 255
	Multiplications	9932	43 621	3391	5151	22 498	28 378

TABLE VI. - RATIOS OF CPU TIMES AND FLOATING-POINT ARITHMETIC OPERATION COUNTS FOR EVALUATING INTEGRALS AS FUNCTIONS OF FORMING THE CHARACTERISTIC ARRAYS FROM LINEAR COMBINATIONS OF THOSE INTEGRALS

[All values refer to the SYMINSE program]

Element designation	Ratio of CPU times*	Ratio of addition counts**	Ratio of multiplication counts**
Parallelogram SQ4	0.286	0.106	0.183
Parallelogram SQ5	.238	.113	.200
Triangle ST6	.142	.00826	.151
Parallelogram SQ8	.0928	.0719	.135
Parallelogram SQ9	.0833	.0767	.146
Triangle ST10	.0725	.0630	.122

*Using data from table III.

**Using data from table V.

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